Geometric Methods: Riemannian Geometry

Recap:  
- Topology = study of shape
- An n-dimensional manifold $M$ is a topological space which "looks" locally like $\mathbb{R}^n$.

Today:  
- Geometry = study of length, angles, volume etc.
- Lengths needed e.g. distance to decision boundary, similarity

In $\mathbb{R}^n$  
\[
\text{d}_{\text{Euclid}}(x, y) = \sqrt{\langle x-y, x-y \rangle_{\text{Euclid}}} \\
\langle u, v \rangle_{\text{Euclid}} = \sum_i u_i v_i
\]

Distance depends on scalar product.

Bilinear Forms

Consider linear transformation $x \rightarrow Ax$ so $x_i = \sum_j A_{ij} \tilde{x}_j$.

For scalar product to remain unchanged
\[
\sum_i x_i y_i = \sum_j (\sum_i A_{ij} \tilde{x}_j) (\sum_k A_{ik} \tilde{y}_k) \\
= \sum_{jk} \tilde{x}_j \tilde{y}_k (\sum_i A_{ij} A_{ik}) \\
= \sum_{ijk} C_{ijk} \tilde{x}_j \tilde{y}_k \quad \text{most general form of scalar product}
\]

Symmetric positive def.
Previously we encountered tangent spaces \( T_pM \)
“space of derivatives at \( p \in M \)”

**Def** A Riemannian Metric \( g_p : T_pM \times T_pM \to \mathbb{R} \)
is a symmetric, positive definite, bilinear form on a tangent space at \( p \in M \).

\[
g_p (X, Y) = \sum_{ij} g_{ij}(p) X^i Y^j
\]

* Metric tensor
  - defined locally & everywhere
  - varies smoothly

* Norm of a tangent vector \( \|X\|_{T_pM} = \sqrt{g_p (X, X)} \)

**Example** Intrinsincs

- Spherical: \( ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \)
- Cartesian: \( ds^2 = dx^2 + dy^2 + dz^2 \)

Equate: \[
\begin{bmatrix}
    dr \\
    d\theta \\
    d\phi
\end{bmatrix}
\begin{bmatrix}
    r^2 \\
    r^2 \sin^2 \theta
\end{bmatrix}
\begin{bmatrix}
    \frac{d}{dt} \\
    \frac{d}{dt}
\end{bmatrix} = \begin{bmatrix}
    \frac{dx}{dt} \\
    \frac{dy}{dt} \\
    \frac{dz}{dt}
\end{bmatrix}
\]

\( g_{ij}(p) \) in spherical coords.
\( g_{ij}(p) \) in cart.

**Def** A Riemannian Manifold \((M, g)\) is a smooth manifold \( M \) equipped with a Riemannian metric \( g \)

**In** Given curve \( \gamma : [0, 1] \to \mathbb{R}^n \)

\[
L(\gamma) = \int_0^1 \| \frac{d\gamma}{dt} \| dt = \int_0^1 \left( \sum_{ij} g_{ij}(\gamma(t)) \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \right)^{1/2} dt
\]

* Length
  - \( \gamma(t) \) velocity time

In \((M, g)\)

\[
L(\gamma) = \int_0^1 \left( \sum_{ij} g_{ij}(\gamma(t)) \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \right)^{1/2} dt
\]
Def. The (geodesic) distance between points \( p, q \in M \) is the length of the shortest curve where \( y(0) = p, y(1) = q \).

\[
d(p, q) = \min \int L(y) \quad \text{for} \quad y(0) = p, y(1) = q
\]

Geodesics generalize straight lines to manifolds.

Usually easier to minimize "energy" \( E(y) := \frac{1}{2} \int_0^1 \left( \sum g_{ij} \frac{dy^i}{dt} \frac{dy^j}{dt} \right) dt \)

Numerical solutions often only optimal, e.g. medical registration

\[
\text{minimize } E(y) = \frac{1}{2} \int_M \phi^* d\mu + \frac{1}{2} \int_0^1 \left| I_0 \phi^*(y) - J(y) \right|^2 dy
\]

Not a good estimate

Follow a geodesic through \( p \) in direction \( K \in T_p M, y(0) = p \).

The exponential map \( \exp_p : T_p M \to M \) is the point \( y(t=1) \) i.e.

\[
\exp_p(K) = y(t=1)
\]

- sometimes invertible
- not always one-to-one
- may not cover \( M \)

Riemann normal coordinates

At \( p \) find basis change to make \( g_{ij}(p) = \delta_{ij} \). \( \exp_p \) does this magically!

At \( p \) find basis change to make \( g_{ij}(p) = \delta_{ij} \). \( \exp_p \) does this magically!
Volumes

What is the volume of $\omega \subset M$?

volume = $\int_\omega d\omega$

Infinitesimal cube:
cat: $d\omega = dx \, dy \, dz$
sph: $d\omega = r^2 \sin \theta \, dr \, d\theta \, d\phi$

$A_{\text{Euclid}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  \quad  $A_{\text{sph}} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \sin^2 \theta \end{bmatrix}$

$\det(A_{\text{Euclid}}) = 1$  \quad  $\det(A_{\text{sph}}) = r^4 \sin^2 \theta$

So maybe

$\text{volume} = \int \sqrt{\det A} \, dx_1 \, dx_2 \, ...$

naturally accounts for changes of basis.

No more pesky Jacobians.